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Irreducibility results for compositions of polynomials in several variables

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Abstract. We obtain explicit upper bounds for the number of irreducible factors for a class of compositions of polynomials in several variables over a given field. In particular, some irreducibility criteria are given for this class of compositions of polynomials.

Keywords. Composition of polynomials; irreducibility results.

1. Introduction

In connection with Hilbert's irreducibility theorem, Cavachi proved in [3] that for any relatively prime polynomials $f(X), g(X) \in \mathbb{Q}[X]$ with $\deg f < \deg g$, the polynomial $f(X) + pg(X)$ is irreducible over $\mathbb{Q}$ for all but finitely many prime numbers p. Sharp explicit upper bounds for the number of factors over $\mathbb{Q}$ of a linear combination $n_1 f(X) + n_2 g(X)$, covering also the case $\deg f = \deg g$, have been derived in [2]. In [1], we realized that by using technics similar to those employed in [4] and [2], upper bounds for the number of factors and irreducibility results can also be obtained for a class of compositions of polynomials of one variable with integer coefficients. More specifically, the following result is proved in [1].

Let $f(X) = a_0 + a_1 X + \cdots + a_m X^m$ and $g(X) = b_0 + b_1 X + \cdots + b_n X^n \in \mathbb{Z}[X]$ be nonconstant polynomials of degree m and n respectively, with $a_0 \neq 0$, and let $L_1(f) = |a_0| + \cdots + |a_{m-1}|$. Assume that d_1 is a positive divisor of a_m and d_2 a positive divisor of b_n such that

$$|a_m| > d_1^{mn} d_2^{m^2 n} L_1(f).$$

Then the polynomial $f \circ g$ has at most $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over $\mathbb{Q}$, where $\Omega(k)$ is the total number of prime factors of k, counting multiplicities. The same conclusion holds in the wider range

$$|a_m| > d_1^{n} d_2^{mn} L_1(f),$$

provided that f is irreducible over $\mathbb{Q}$.

In the present paper we provide explicit upper bounds for the number of factors, and irreducibility results for a class of compositions of polynomials in several variables over a given field. We will deduce this result from the corresponding result for polynomials in two variables X,Y over a field K. We use the following notation. For any polynomial $f \in K[X,Y]$ we denote by $\deg_Y f$ the degree of f as a polynomial in Y, with coefficients in $K[X]$. Then we write any polynomial $f \in K[X,Y]$ in the form

$$f = a_0(X) + a_1(X)Y + \cdots + a_d(X)Y^d,$$

with $a_0, a_1, \ldots, a_d$ in $K[X]$, $a_d \neq 0$, and define

$$H_1(f) = \max\{\deg a_0, \ldots, \deg a_{d-1}\}.$$

Finally, for any polynomial $f \in K[X]$ we denote by $\Omega(f)$ the number of irreducible factors of f, counting multiplicities ($\Omega(c) = 0$ for $c \in K$). We will prove the following theorem.

Theorem 1. *Let K be a field and let $f(X,Y) = a_0 + a_1Y + \cdots + a_mY^m$, $g(X,Y) = b_0 + b_1Y + \cdots + b_nY^n$, with $a_0, a_1, \ldots, a_m$, $b_0, b_1, \ldots, b_n \in K[X]$, $a_0a_mb_n \neq 0$. If d_1 is a factor of a_m and d_2 a factor of b_n such that*

$$\deg a_m > mn\deg d_1 + m^2 n\deg d_2 + H_1(f),$$

then the polynomial $f(X, g(X,Y))$ has at most $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over $K(X)$. The same conclusion holds in the wider range

$$\deg a_m > n\deg d_1 + mn\deg d_2 + H_1(f),$$

provided that f is irreducible over $K(X)$.

Theorem 1 provides, in particular, bounds for the number of irreducible factors of $f(X,Y)$ over $K(X)$, by taking $g(X,Y) = Y$.

COROLLARY 1.

Let K be a field and let $f(X,Y) = a_0 + a_1Y + \cdots + a_mY^m$, with $a_0, a_1, \ldots, a_m \in K[X]$, $a_0a_m \neq 0$. If d is a factor of a_m such that

$$\deg a_m > m\deg d + H_1(f),$$

then the polynomial $f(X,Y)$ has at most $\Omega(a_m/d)$ irreducible factors over $K(X)$.

Under the assumption that a_m has an irreducible factor over K of large enough degree, we have the following irreducibility criteria.

COROLLARY 2.

Let K be a field and let $f(X,Y) = a_0 + a_1Y + \cdots + a_mY^m$, with $a_0, a_1, \ldots, a_m \in K[X]$, $a_0a_m \neq 0$. If $a_m = pq$ with $p, q \in K[X]$, p irreducible over K, and

$$\deg p > (m-1)\deg q + H_1(f),$$

then the polynomial $f(X,Y)$ is irreducible over $K(X)$.

COROLLARY 3.

Let K be a field and let $f(X,Y) = a_0 + a_1Y + \cdots + a_mY^m$, $g(X,Y) = b_0 + b_1Y + \cdots + b_nY^n$, with $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in K[X]$, $a_0a_mb_n \neq 0$, and f irreducible over $K(X)$. If $a_m = pq$ with $p, q \in K[X]$, p irreducible over K, and

$$\deg p > (n-1)\deg q + mn\deg b_n + H_1(f),$$

then the polynomial $f(X, g(X,Y))$ is irreducible over $K(X)$.

COROLLARY 4.

Let K be a field and let $f(X,Y) = a_0 + a_1Y + \cdots + a_mY^m$, $g(X,Y) = b_0 + b_1Y + \cdots + b_nY^n$, with $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in K[X]$, $a_0a_mb_n \neq 0$. If $a_m = pq$ with $p, q \in K[X]$, p irreducible over K, and

$$\deg p > \max\{(m-1)\deg q,\ (n-1)\deg q + mn\deg b_n\} + H_1(f),$$

then the polynomial $f(X, g(X,Y))$ is irreducible over $K(X)$.

Another consequence of Theorem 1 is the following corresponding result for polynomials in $r \geq 2$ variables $X_1, X_2, \ldots, X_r$ over K. In this case, for any polynomial $f \in K[X_1, \ldots, X_r]$, $\Omega(f)$ will stand for the number of irreducible factors of f over $K(X_1, \ldots, X_{r-1})$, counting multiplicities. Then, for any polynomial $f \in K[X_1, \ldots, X_r]$ and any $j \in \{1, \ldots, r\}$ we denote by $\deg_{X_j} f$ the degree of f as a polynomial in X_j with coefficients in $K[X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_r]$. We also write any polynomial $f \in K[X_1, \ldots, X_r]$ in the form

$$\begin{aligned} f = {} & a_0(X_1, \ldots, X_{r-1}) + a_1(X_1, \ldots, X_{r-1})X_r \\ & + \cdots + a_d(X_1, \ldots, X_{r-1})X_r^d, \end{aligned}$$

with $a_0, a_1, \ldots, a_d \in K[X_1, \ldots, X_{r-1}]$, $a_0 \neq 0$, and for any $j \in \{1, \ldots, r-1\}$ we let

$$H_j(f) = \max\{\deg_{X_j} a_0, \deg_{X_j} a_1, \ldots, \deg_{X_j} a_{d-1}\}.$$

Then one has the following result.

COROLLARY 5.

Let K be a field, $r \geq 2$, and let $f(X_1, \ldots, X_r) = a_0 + a_1X_r + \cdots + a_mX_r^m$, $g(X_1, \ldots, X_r) = b_0 + b_1X_r + \cdots + b_nX_r^n$, with $a_0, a_1, \ldots, a_m$, $b_0, b_1, \ldots, b_n \in K[X_1, \ldots, X_{r-1}]$, $a_0a_mb_n \neq 0$. If d_1 is a factor of a_m and d_2 a factor of b_n such that for an index $j \in \{1, \ldots, r-1\}$,

$$\deg_{X_j} a_m > mn\deg_{X_j} d_1 + m^2n\deg_{X_j} d_2 + H_j(f),$$

then the polynomial $f(X_1, \ldots, X_{r-1}, g(X_1, \ldots, X_r))$ has at most $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over the field $K(X_1, \ldots, X_{r-1})$. The same conclusion holds in the wider range

$$\deg_{X_j} a_m > n\deg_{X_j} d_1 + mn\deg_{X_j} d_2 + H_j(f),$$

provided that f is irreducible over $K(X_1, \ldots, X_{r-1})$.

In particular we have the following irreducibility criterion.

COROLLARY 6.

Let K be a field, $r \geq 2$, and let $f(X_1,\dots,X_r) = a_0 + a_1X_r + \cdots + a_mX_r^m$, $g(X_1,\dots,X_r) = b_0 + b_1X_r + \cdots + b_nX_r^n$, with $a_0,a_1,\dots,a_m,b_0,b_1,\dots,b_n \in K[X_1,\dots,X_{r-1}], a_0a_mb_n \neq 0$. If $a_m = p \cdot q$ with p a prime element of the ring $K[X_1,\dots,X_{r-1}]$ such that for an index $j \in \{1,\dots,r-1\}$,

$$\deg_{X_j} p > \max\{(m-1)\deg_{X_j} q, (n-1)\deg_{X_j} q + mn\deg_{X_j} b_n\} + H_j(f),$$

then the polynomial $f(X_1,\dots,X_{r-1},g(X_1,\dots,X_r))$ is irreducible over the field $K(X_1,\dots,X_{r-1})$.

Corollary 5 follows from Theorem 1 by writing Y for X_r and X for X_j, where j is any index for which

$$\deg_{X_j} a_m > mn\deg_{X_j} d_1 + m^2n\deg_{X_j} d_2 + H_j(f),$$

and by replacing the field K with the field generated by K and the variables $X_1,X_2,\dots,$ X_{r-1} except for X_j.

The reader may naturally wonder how sharp the above results are. In this connection, we discuss a couple of examples in the next section.

2. Examples

Let $K = \mathbb{Q}$, choose integers $m,d \geq 2$, select polynomials $a_0(X),a_1(X),\dots,a_{m-1}(X) \in \mathbb{Q}[X]$ with $a_0(X) \neq 0$, and consider the polynomial in two variables $f(X,Y)$ given by

$$f(X,Y) = a_0(X) + a_1(X)Y + \cdots + a_{m-1}(X)Y^{m-1} + (X^d + 5X + 5)Y^m.$$

Under these circumstances, in terms of the degrees of the polynomials $a_0(X),a_1(X),$ $\dots,a_{m-1}(X)$, can we be sure that the polynomial $f(X,Y)$ is irreducible over $\mathbb{Q}(X)$? The polynomial $p(X) = X^d + 5X + 5$ is an Eisensteinian polynomial with respect to the prime number 5, and hence it is irreducible over $\mathbb{Q}$. We may then apply Corollary 2, with $q = 1$, in order to conclude that $f(X,Y)$ is irreducible over $\mathbb{Q}(X)$ as long as $H_1(f) < d$, that is, as long as each of the polynomials $a_0(X)$, $a_1(X)$, $\dots,a_{m-1}(X)$ has degree less than or equal to $d-1$. We remark that for any choice of $m,d \geq 2$ this bound is the best possible, in the sense that there are polynomials $a_0(X),a_1(X),\dots,a_{m-1}(X) \in \mathbb{Q}[X]$, $a_0(X) \neq 0$, for which

$$\max\{\deg_X a_0(X), \deg_X a_1(X), \dots, \deg_X a_{m-1}(X)\} = d,$$

such that the corresponding polynomial $f(X,Y)$ is reducible over $\mathbb{Q}(X)$. Indeed, one may choose for instance $a_0(X),a_1(X),\dots,a_{m-2}(X)$ to be any polynomials with coefficients in $\mathbb{Q}$, with $a_0(X) \neq 0$, of degrees less than or equal to d, and define $a_{m-1}(X)$ by the equality

$$a_{m-1}(X) = -X^d - 5X - 5 - \sum_{0 \leq i \leq m-2} a_i(X).$$

Then, on the one hand, we will have $\max\{\deg_X a_0,\dots,\deg_X a_{m-1}\} = d$ and on the other hand, the corresponding polynomial $f(X,Y)$ will be reducible over $\mathbb{Q}(X)$, being divisible by $Y-1$.

In the next example, let us slightly modify the polynomial $f(X,Y)$, and choose a polynomial $g(X,Y)$ of arbitrary degree, say

$$f(X,Y) = a_0 + a_1Y + \cdots + a_{m-1}Y^{m-1} + (X^n + 5X + 5)^2Y^m,$$
$$g(X,Y) = b_0 + b_1Y + \cdots + b_{n-1}Y^{n-1} + Y^n,$$

where $a_0, a_1, \ldots, a_{m-1},\ b_0, b_1, \ldots, b_{n-1} \in \mathbb{Q}[X]$, $a_0(X) \neq 0$. We may apply Theorem 1, with $d_1 = d_2 = 1$, in order to conclude that $f(X,g(X,Y))$ has at most two irreducible factors over $\mathbb{Q}(X)$ as long as $H_1(f) < 2n$, that is, as long as each of the polynomials $a_0, a_1, \ldots, a_{m-1}$ has degree less than or equal to $2n-1$. This bound too is the best possible, as there exist polynomials $a_0, a_1, \ldots, a_{m-1} \in \mathbb{Q}[X]$, $a_0(X) \neq 0$, $g \in \mathbb{Q}[Y]$, g monic, for which

$$\max\{\deg a_0, \deg a_1, \ldots, \deg a_{m-1}\} = 2n,$$

such that the corresponding polynomial $f(X,g(X,Y))$ has at least three irreducible factors over $\mathbb{Q}(X)$. For instance, one may take $g(Y) = Y^2$, choose polynomials $a_0(X), a_1(X), \ldots, a_{m-2}(X)$ with coefficients in $\mathbb{Q}$, with $a_0(X) \neq 0$, of degrees less than or equal to $2n$, and define $a_{m-1}(X)$ by the equality

$$a_{m-1}(X) = -(X^n + 5X + 5)^2 - \sum_{0 \leq i \leq m-2} a_i(X).$$

Then we will have $\max\{\deg a_0, \ldots, \deg a_{m-1}\} = 2n$, while the corresponding polynomial $f(X,g(X,Y))$ will have at least three irreducible factors over $\mathbb{Q}(X)$, being divisible by $Y^2 - 1$.

3. Proof of Theorem 1

Let K, $f(X,Y)$, $g(X,Y)$, d_1 and d_2 be as in the statement of the theorem. Let $b \in K[X]$ denote the greatest common divisor of $b_0, b_1, \ldots, b_n$, and define the polynomial $\overline{g}(X,Y) \in K[X,Y]$ by the equality

$$g(X,Y) = b_0 + b_1Y + \cdots + b_nY^n = b\overline{g}(X,Y).$$

Next, let $a \in K[X]$ denote the greatest common divisor of the coefficients of $f(X,g(X,Y))$ viewed as a polynomial in Y, and define the polynomial $F(X,Y) \in K[X,Y]$ by the equality

$$f(X,g(X,Y)) = aF(X,Y).$$

If we assume that $f(X,g(X,Y))$ has $s > \Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over $K(X)$, then the polynomial $F(X,Y)$ will have a factorization $F(X,Y) = F_1(X,Y)\cdots F_s(X,Y)$, with $F_1(X,Y), \ldots, F_s(X,Y) \in K[X,Y]$, $\deg_Y F_1(X,Y) \geq 1, \ldots, \deg_Y F_s(X,Y) \geq 1$. Let $t_1, \ldots, t_s \in K[X]$ be the leading coefficients of $F_1(X,Y), \ldots, F_s(X,Y)$ respectively, viewed as polynomials in Y. By comparing the
leading coefficients in the equality

$$a_0 + a_1g(X,Y) + \cdots + a_mg^m(X,Y) = aF_1(X,Y)\cdots F_s(X,Y)$$

we obtain the following equality in $K[X]$:

$$at_1\cdots t_s = a_m b_n^m = d_1 d_2^m \cdot \frac{a_m}{d_1} \cdot \left(\frac{b_n}{d_2}\right)^m. \tag{1}$$

Then, in view of (1), it follows easily that at least one of the t_i's, say t_1, will divide $d_1 d_2^m$. As a consequence, one has

$$\deg t_1 \le \deg d_1 + m \deg d_2. \tag{2}$$

We now consider the polynomial

$$\begin{aligned} h(X,Y) &= f(X,g(X,Y)) - a_m(X)g(X,Y)^m \\ &= a_0(X) + a_1(X)g(X,Y) + \cdots + a_{m-1}(X)g(X,Y)^{m-1}. \end{aligned}$$

Recall that a_0 and $\overline{g}$ are relatively prime, and $a_0(X) \neq 0$. It follows that the polynomials $\overline{g}^m(X,Y)$ and $h(X,Y)$ are relatively prime. Therefore $\overline{g}^m(X,Y)$ and $F_1(X,Y)$ are relatively prime. As a consequence, the resultant $R(\overline{g}^m, F_1)$ of $\overline{g}^m(X,Y)$ and $F_1(X,Y)$, viewed as polynomials in Y with coefficients in $K[X]$, will be a nonzero element of $K[X]$. We now introduce a nonarchimedean absolute value $|\cdot|$ on $K(X)$, as follows. We fix a real number ρ, with $0<\rho<1$, and for any polynomial $F(X) \in K[X]$ we define $|F(X)|$ by the equality

$$|F(X)| = \rho^{-\deg F(X)}. \tag{3}$$

We then extend the absolute value $|\cdot|$ to $K(X)$ by multiplicativity. Thus for any $L(X) \in K(X)$, $L(X) = \frac{F(X)}{G(X)}$, with $F(X), G(X) \in K[X]$, $G(X) \neq 0$, let $|L(X)| = \frac{|F(X)|}{|G(X)|}$. Let us remark that for any non-zero element u of $K[X]$ one has $|u| \ge 1$. In particular, $R(\overline{g}^m, F_1)$ being a non-zero element of $K[X]$, we have

$$|R(\overline{g}^m, F_1)| \ge 1. \tag{4}$$

Next, we estimate $|R(\overline{g}^m, F_1)|$ in a different way. Let $\overline{K(X)}$ be a fixed algebraic closure of $K(X)$, and let us fix an extension of the absolute value $|\cdot|$ to $\overline{K(X)}$, which we will also denote by $|\cdot|$. Consider now the factorizations of $\overline{g}(X,Y)$, $\overline{g}^m(X,Y)$ and $F_1(X,Y)$ over $\overline{K(X)}$. Say

$$\begin{aligned} \overline{g}(X,Y) &= \overline{b}_n(Y-\xi_1)\cdots(Y-\xi_n), \\ \overline{g}^m(X,Y) &= \overline{b}_n^m(Y-\xi_1)^m\cdots(Y-\xi_n)^m \end{aligned}$$

and

$$F_1(X,Y) = t_1(Y-\theta_1)\cdots(Y-\theta_r),$$

with $\xi_1,\ldots,\xi_n,\theta_1,\ldots,\theta_r \in \overline{K(X)}$. Here $1 \le r \le mn-1$, by our assumption that $\deg_Y F_1(X,Y) \ge 1$ and $\deg_Y F_2(X,Y) \ge 1$. Then

$$|R(\overline{g}^m, F_1)| = \left|\overline{b}_n^{mr} t_1^{mn} \prod_{1\le i\le n}\prod_{1\le j\le r}(\xi_i-\theta_j)^m\right| = |t_1|^{mn}\prod_{1\le j\le r}|\overline{g}^m(X,\theta_j)|. \tag{5}$$

The fact that $F_1(X,\theta_j)=0$ for any $j\in\{1,\dots,r\}$ also implies that $f(X,g(X,\theta_j))=0$, and so

$$\begin{aligned} h(X,\theta_j) &= f(X,g(X,\theta_j)) - a_m(X)g^m(X,\theta_j) \\ &= -a_m(X)b^m(X)\overline{g}^m(X,\theta_j). \end{aligned}$$

Since $b(X)$ is a non-zero element of $K[X]$, one has $|b(X)|\geq 1$. We deduce that

$$|\overline{g}^m(X,\theta_j)| = \frac{|h(X,\theta_j)|}{|a_m(X)b^m(X)|} \leq \frac{|h(X,\theta_j)|}{|a_m(X)|}. \tag{6}$$

By combining (5) and (6) we find that

$$|R(\overline{g}^m,F_1)| \leq \frac{|t_1|^{mn}}{|a_m|^r}\prod_{1\leq j\leq r}|h(X,\theta_j)|. \tag{7}$$

We now proceed to find an upper bound for $|h(X,\theta_j)|$, for $1\leq j\leq r$. In order to do this, we first use the identity

$$h(X,Y) = a_0(X)+a_1(X)g(X,Y)+\cdots+a_{m-1}(X)g(X,Y)^{m-1}$$

to obtain

$$\begin{aligned} |h(X,\theta_j)| &= |a_0(X)+a_1(X)g(X,\theta_j)+\cdots+a_{m-1}(X)g(X,\theta_j)^{m-1}| \\ &\leq \max_{0\leq k\leq m-1}|a_k(X)|\cdot|g(X,\theta_j)|^k, \end{aligned} \tag{8}$$

for $1\leq j\leq r$. Next, we consider the factorization of $f(X,Y)$ over $\overline{K(X)}$, say

$$f(X,Y) = a_m(X)(Y-\lambda_1)\cdots(Y-\lambda_m),$$

with $\lambda_1,\dots,\lambda_m\in\overline{K(X)}$. For any $i\in\{1,\dots,m\}$ one has

$$0 = f(X,\lambda_i) = a_0(X)+a_1(X)\lambda_i+\cdots+a_m(X)\lambda_i^m. \tag{9}$$

By (9) we see that

$$\begin{aligned} |a_m(X)|\cdot|\lambda_i^m| &= |a_0(X)+a_1(X)\lambda_i+\cdots+a_{m-1}(X)\lambda_i^{m-1}| \\ &\leq \max_{0\leq c\leq m-1}|a_c(X)|\cdot|\lambda_i|^c. \end{aligned} \tag{10}$$

For any $i\in\{1,\dots,m\}$ let us select an index $c_i\in\{0,\dots,m-1\}$ for which the maximum is attained on the right side of (10). We then have $|a_m(X)|\cdot|\lambda_i|^m\leq|a_{c_i}(X)|\cdot|\lambda_i|^{c_i}$, and so

$$|\lambda_i| \leq \left(\frac{|a_{c_i}(X)|}{|a_m(X)|}\right)^{1/(m-c_i)}. \tag{11}$$

We now return to (8). Fix a $j\in\{1,\dots,r\}$. In order to provide an upper bound for $|h(X,\theta_j)|$, it is sufficient to find an upper bound for $|g(X,\theta_j)|$. Recall that $f(X,g(X,\theta_j))=0$. Therefore there exists an $i\in\{1,\dots,m\}$, depending on j, for which $g(X,\theta_j)=\lambda_i$. Then, by (11) we obtain

$$|g(X,\theta_j)| \leq \left(\frac{|a_{c_i}(X)|}{|a_m(X)|}\right)^{1/(m-c_i)} \leq \max_{1\leq v\leq m}\left(\frac{|a_{m-v}(X)|}{|a_m(X)|}\right)^{1/v}. \tag{12}$$

Inserting (12) in (8) we conclude that, uniformly for $1 \le j \le r$, one has

$$|h(X,\theta_j)| \le \max_{\substack{1\le v\le m\\ 0\le k\le m-1}} |a_k| \left(\frac{|a_{m-v}|}{|a_m|}\right)^{k/v}. \tag{13}$$

Combining (13) with (7) we derive the inequality

$$|R(\overline{g}^m, F_1)| \le \frac{|t_1|^{mn}}{|a_m|^r} \max_{\substack{1\le v\le m\\ 0\le k\le m-1}} \frac{|a_k|^r \cdot |a_{m-v}|^{rk/v}}{|a_m|^{rk/v}},$$

which may be written as

$$|R(\overline{g}^m, F_1)| \le |t_1|^{mn} \left(\max_{\substack{1\le v\le m\\ 0\le k\le m-1}} \frac{|a_k| \cdot |a_{m-v}|^{k/v}}{|a_m|^{1+k/v}} \right)^r. \tag{14}$$

In what follows we are going to prove that

$$|t_1|^{mn} \max_{\substack{1\le v\le m\\ 0\le k\le m-1}} \frac{|a_k| \cdot |a_{m-v}|^{k/v}}{|a_m|^{1+k/v}} < 1, \tag{15}$$

which by (14) will contradict (4), since $r \ge 1$. Using the definition of the absolute value $|\cdot|$, we write the inequality (15) in the form

$$\max_{\substack{1\le v\le m\\ 0\le k\le m-1}} \rho^{(1+\frac{k}{v})\deg a_m - \deg a_k - \frac{k}{v}\deg a_{m-v}} < \rho^{mn \deg_X t_1},$$

which is equivalent to

$$\min_{\substack{1\le v\le m\\ 0\le k\le m-1}} \left\{ \left(1+\frac{k}{v}\right)\deg a_m - \deg a_k - \frac{k}{v}\deg a_{m-v} \right\} > mn \deg t_1. \tag{16}$$

By combining (16) with (2), it will be sufficient to prove that

$$\min_{\substack{1\le v\le m\\ 0\le k\le m-1}} \left\{ \left(1+\frac{k}{v}\right)\deg a_m - \deg a_k - \frac{k}{v}\deg a_{m-v} \right\}$$
$$> mn \deg d_1 + m^2 n \deg d_2,$$

or equivalently,

$$\min_{\substack{1\le v\le m\\ 0\le k\le m-1}} \left(1+\frac{k}{v}\right) \cdot \left\{ \deg a_m - \frac{\deg a_k + \frac{k}{v}\deg a_{m-v}}{1+\frac{k}{v}} \right\}$$
$$> mn \deg d_1 + m^2 n \deg d_2. \tag{17}$$

By our assumption on the size of $\deg a_m$ we have

$$\deg a_m - H_1(f) > mn \deg d_1 + m^2 n \deg d_2,$$

from which (17) follows, since

$$\max_{\substack{1\le v\le m\\ 0\le k\le m-1}} \frac{\deg a_k + \frac{k}{v}\deg a_{m-v}}{1+\frac{k}{v}} \le H_1(f). \tag{18}$$

This completes the proof of the first part of the theorem. Assuming now that f is irreducible over $K(X)$, the proof goes as in the first part, except that now we have $\deg_Y F_1 = r \ge m$, since by Capelli's Theorem [5], the degree in Y of every irreducible factor of $f(X,g(X,Y))$ must be a multiple of m. Therefore, instead of (15) one has to prove that

$$|t_1|^n \max_{\substack{1\le v\le m\\ 0\le k\le m-1}} \frac{|a_k|\cdot|a_{m-v}|^{k/v}}{|a_m|^{1+k/v}} < 1,$$

which is equivalent to

$$\min_{\substack{1\le v\le m\\ 0\le k\le m-1}} \left\{\left(1+\frac{k}{v}\right)\deg a_m - \deg a_k - \frac{k}{v}\deg a_{m-v}\right\} > n\deg t_1.$$

By combining this inequality with (2), it will be sufficient to prove that

$$\min_{\substack{1\le v\le m\\ 0\le k\le m-1}} \left(1+\frac{k}{v}\right)\cdot\left\{\deg a_m - \frac{\deg a_k + \frac{k}{v}\deg a_{m-v}}{1+\frac{k}{v}}\right\}$$
$$> n\deg d_1 + mn\deg d_2. \tag{19}$$

Finally, our assumption that $\deg a_m - H_1(f) > n\deg d_1 + mn\deg d_2$ together with (18) imply (19), which completes the proof of the theorem. □

We end by noting that in the statement of Theorem 1, the assumption on the size of $\deg a_m$, and the bound $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ exhibited for the number of factors do not depend on the first n coefficients of g. So these bounds remain the same once we fix $n, b_n(X)$ and $d_2(X)$, and let $b_0(X),\ldots,b_{n-1}(X)$ vary independently.

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References

[1] Bonciocat A I and Zaharescu A, Irreducibility results for compositions of polynomials with integer coefficients, preprint.

[2] Bonciocat N C, Upper bounds for the number of factors for a class of polynomials with rational coefficients, *Acta Arith.* **113(2)** (2004) 175–187

[3] Cavachi M, On a special case of Hilbert's irreducibility theorem, *J. Number Theory* **82(1)** (2000) 96–99

[4] Cavachi M, Vâjâitu M and Zaharescu A, A class of irreducible polynomials, *J. Ramanujan Math. Soc.* **17(3)** (2002) 161–172

[5] Schinzel A, Polynomials with Special Regard to Reducibility, in: Encyclopedia of Mathematics and its Applications (Cambridge University Press) (2000)